SEMINAR ON WORK OF WEISS AND WILLIAMS

MANUEL KRANNICH

This seminar discusses some of the contributions to geometric topology by Michael Weiss and Bruce Williams.

1. The stable Hatcher spectral sequence—spacified

The goal of the first part of the seminar is to understand the following result of [WW88].

Theorem A (Weiss–Williams). *For* Cat $\in$ {Diff, Top} *and M a compact* Cat*-manifold, there exists a $\mathbf{Z}/2$-spectrum F_M such that*

(i) $\Omega^{\infty+1}F_M$ *agrees with the space $C(M \times D^\infty)$ of stable* Cat*-concordances of M, and*
(ii) there exists a map

$$\widetilde{\mathrm{Cat}}_\partial(M)/\mathrm{Cat}_\partial(M) \longrightarrow \Omega^\infty(F_M\langle 0\rangle_{h\mathbf{Z}/2})$$

which is $(\phi(M)+1)$-connected, where $\phi(-)$ is the Cat*-concordance stable range.*

Remark.

(i) Note that Theorem A is independent of the connectivity and dimension of M.
(ii) In [WW88], the spectrum F_M is denoted $\Omega\underline{\underline{\mathrm{Wh}}}(M)$, referring to the (loop space of the) Cat-Whitehead spectrum of M. We prefer to avoid this notation, since the proof of Theorem A does not rely on the stable parametrised h-cobordism theorem connecting $C(M \times D^\infty)$ to the usual Whitehead spectrum.
(iii) The homotopy orbit spectral sequence of $F_M\langle 0\rangle_{h\mathbf{Z}/2}$ has the form

$$E^2_{p,q} \cong \mathrm{H}_p(\mathbf{Z}/2; \pi_{q-1}(C(M)) \implies \pi_{p+q}(\widetilde{\mathrm{Cat}}_\partial(M)/\mathrm{Cat}_\partial(M))$$

in the range $p+q \leq \phi(M)$, so Theorem A could be considered as a space-level version of the classical Hatcher spectral sequence, at least in the stable range.[1]

As in [WW88], we shall mainly focus on the case Cat = Top.[2] Weiss–Williams deal with various technicalities by using different categories of spaces (topological spaces, simplicial sets, and their very own *virtual spaces*), but we shall ignore any issues of this sort. Numbered items in [WW88] are referred to by WW.0.1, WW.0.2, etc.

Outline of proof. In Talks 1 and 2, we construct a spectrum F_M in terms of bounded homeomorphisms of M and establish some of its properties. We will see that its nth space $(F_M)_n = \mathrm{Top}^b(M\times\mathbf{R}^{n+1})/\mathrm{Top}^b(M\times\mathbf{R}^n)$ is an $(n+1)$-fold delooping of the space $C(M\times D^n)$ of concordances of $M \times D^n$: there is an equivalence

$$\Omega^{n+1}(\mathrm{Top}^b(M\times\mathbf{R}^{n+1})/\mathrm{Top}^b(M\times\mathbf{R}^n)) \simeq C(M\times D^n) \tag{1}$$

which stabilises to

$$\Omega^{\infty+1}F_M \simeq C(M\times D^\infty). \tag{2}$$

In Talks 3 and 4, we introduce a geometric involution on F_M and construct maps

$$\mathrm{Top}^b(M\times\mathbf{R}^n)/\mathrm{Top}(M) \to \Gamma(F_M^{tw}(T\mathbf{R}P^n)) \to \Omega^\infty(S^n_+ \wedge_{\mathbf{Z}/2} F_M), \tag{3}$$

[1]Note that it is not obvious that these two spectral sequences agree in this range.
[2]Some of the necessary adjustments to Cat = Diff are explained in WW.1.17.

the first being the so-called *hyperplane test* and the second a twisted Poincaré duality map. Their composition stabilises to a map

$$\mathrm{Top}^b(M\times \mathbf{R}^\infty)/\mathrm{Top}(M) \to \Omega^\infty(S^\infty_+ \wedge_{\mathbf{Z}/2} F_M) \simeq \Omega^\infty((F_M)_{h\mathbf{Z}/2}) \tag{4}$$

and fits for varying n into a map of fibre sequences

$$\begin{tikzcd} \mathrm{Top}^b(M\times \mathbf{R}^n)/\mathrm{Top}(M) \arrow[r] \arrow[d] & \Omega^\infty(S^{n-1}_+ \wedge_{\mathbf{Z}/2} F_M) \arrow[d] \\ \mathrm{Top}^b(M\times \mathbf{R}^{n+1})/\mathrm{Top}(M) \arrow[r] \arrow[d] & \Omega^\infty(S^{n}_+ \wedge_{\mathbf{Z}/2} F_M) \arrow[d] \\ (F_M)_n \arrow[r] & \Omega^{\infty-n} F_M \end{tikzcd} \tag{5}$$

whose bottom map, looped $(n+1)$ times, agrees with respect to (1) with the stabilisation map $C(M\times D^n)\to C(M\times D^\infty)$. The rest of the argument proceeds roughly as follows:[3]

a) Show that $(F_M)_n$ and $\Omega^{\infty-n}F_M$ are n-connected, so the bottom horizontal map in (5) is $(\phi(M)+n+1)$-connected. This implies that the middle horizontal map is $(\phi(M)+1)$-connected by induction and hence so is the stabilised map (4).
b) Show that the natural map $\mathrm{Top}^b(M\times\mathbf{R}^\infty)\to\widetilde{\mathrm{Top}}^b(M\times\mathbf{R}^\infty)$ is an equivalence.
c) Prove that $\widetilde{\mathrm{Top}}(M)\to\widetilde{\mathrm{Top}}^b(M\times\mathbf{R}^\infty)$ is an equivalence as well, which concludes the proof of Theorem A.

Step b) is being dealt with as part of Talk 4. To achieve a) and c), Weiss and Williams offer two different strategies. If one is willing to restrict to simply connected manifolds of dimension ≥ 5, then a) and c) can be proved using the *bounded h-cobordism theorem* and some algebraic K-theory. Alternatively, there is a more direct proof based on various simplicial arguments that do not rely on K-theory and work for all manifolds. We focus on the first approach, using bounded K-theory.

Talk 0: Introduction and overview (MK, 10/10, 14–15, MR 13).

Talk 1: The belt buckle trick (MB, 16/10, 14–15, MR 15). Introduce the spaces $\mathrm{Top}^b(M\times\mathbf{R}^k)$ of *bounded homeomorphisms* for a compact manifold M (see WW.0.4), explain the *belt buckle trick* of WW.1.2, and use it to establish the delooping (see WW.1.5)

$$\mathrm{Top}^b(M\times D^k\times\mathbf{R}^n)\simeq\Omega^k\,\mathrm{Top}^b(M\times\mathbf{R}^{n+k}). \tag{6}$$

Talk 2: The spectrum F_M and its relation to concordances (OR-W, 24/10, 14–15, MR 13). Define F_M as a (sequential) spectrum with nth space $\mathrm{Top}^b(M\times\mathbf{R}^{n+1})/\mathrm{Top}^b(M\times\mathbf{R}^n)$ (see WW.1.11), and use the delooping (6) to relate ΩF_M to the spaces $C^b(M\times\mathbf{R}^n)$ of *bounded concordances* of $M\times\mathbf{R}^n$ by proving WW.1.8 and WW.1.10, which show

$$\begin{gathered}\Omega(\mathrm{Top}^b(M\times\mathbf{R}^{n+1})/\mathrm{Top}^b(M\times\mathbf{R}^n))\simeq C^b(M\times\mathbf{R}^n)\\ \Omega^nC^b(M\times\mathbf{R}^n\times\mathbf{R}^k)\simeq C^b(M\times D^n\times\mathbf{R}^k)\end{gathered} \tag{7}$$

and in particular establish (1). Show furthermore that the first equivalence in (7) is compatible with the stabilisation maps on both sides (see WW.1.12) and conclude (2).

[3]If M is not simply connected or of dimension ≤ 4, this is not quite true and some modifications are necessary.

Talk 3: Coordinate free spectra and a Poincaré duality map (MB, 31/10, 14–15, MR 13). Recall the notion of a coordinate free spectrum F with involution (see WW.2.1 and WW.2.3) and explain the Poincaré duality stabilisation map $\Gamma(F(TW)) \to \Omega^\infty(W/\partial W \wedge F)$ (see WW.2.4), where $\Gamma(F(TW))$ is the space of sections of the bundle obtained by fibrewise applying F to the tangent bundle of a given smooth manifold W. Discuss the naturality of this map in W (see WW.2.6 and WW.2.7) and mention its $\mathbf{Z}/2$-twisted analogue $\Gamma(F^{tw}(TW)) \to \Omega^\infty(\widetilde{W}/\widetilde{\partial W} \wedge_{\mathbf{Z}/2} F)$ (see WW.2.8) for double covers $\widetilde{W} \to W$ and spectra F with involution.

Talk 4: The hyperplane test (MK, 7/11, 14–15, MR 13). Observe that F_M of Talk 2 is naturally a coordinate free spectrum as introduced in the Talk 3 and introduce its canonical involution (see WW.1.11 and the beginning of WW.3). Discuss the *hyperplane test*

$$\mathrm{Top}^b(M \times \mathbf{R}^{n+1})/\mathrm{Top}(M) \to \Gamma(F_M^{tw}(T\mathbf{R}P^n))$$

of WW.3.1, combine it with the Poincaré duality map of the previous talk to the map (3), and prove WW.3.2 in order to stabilise it to (4). Continue by explaining the map (5) of fibre sequences (see WW.3.4) and note that the $(n+1)$st looping of the bottom map agrees with the stabilisation map $C(M \times D^n) \to C(M \times D^\infty)$. Finish the talk with step b) above by making good for the missing WW.1.14.

Talk 5: Interlude—Bounded geometry and negative algebraic K-theory (NP, 11/11, 13–14, MR 6). This talk is dedicated to the *bounded h-cobordism theorem*. Explain its statement as in WW.5.1 and give an idea of its proof following [Ped86], which is parallel to the proof of the usual s-cobordism theorem if one uses Pedersen's description of negative K-groups in terms of *bounded K-theory* (see [Ped84b, Ped84a]).

Talk 6: The negative homotopy groups of F_M (NP, 21/11, 14–15, MR 13). Use the bounded h-cobordism theorem to identify the homotopy groups of the nth space of the spectrum, $\mathrm{Top}^b(M \times \mathbf{R}^{n+1})/\mathrm{Top}^b(M \times \mathbf{R}^n)$ with negative K-groups of $\mathbf{Z}[\pi_1 M]$ in the range $0 < * \le n$ by proving WW.5.3 and combining it with (6). Explain what we learn about degree 0 from this approach (see WW.5.4). Show that the homotopy groups of the block analogue $\widetilde{\mathrm{Top}}^b(M \times \mathbf{R}^{n+1})/\widetilde{\mathrm{Top}}^b(M \times \mathbf{R}^n)$ can also be expressed in terms of K-theory, even without restriction on the degree (WW.5.5). Use this, together with [Ped84b] and [BHS64] (which we treat as a black box) to conclude a) and c) for simply connected M of dimension at least 5 (see WW.5.7). This finishes the proof of Theorem A for Cat = Top and simply connected M of dimension at least 5.

Talk 7: A different perspective (ORW, 05/12, 14–15, MR 13). Give some idea of WW.3.7., which provides a different perspective on the map

$$\widetilde{\mathrm{Cat}}_\partial(M)/\mathrm{Cat}_\partial(M) \to \Omega^\infty(F_M\langle 0\rangle_{h\mathbf{Z}/2}),$$

not using bounded automorphisms. If time permits, try to give an idea of the closely related short (but dense) work of Weiss [Wei86][4] on the *Gromoll filtration* of the group of exotic spheres Θ_n. The connection goes roughly as follows: Note that $\mathrm{Diff}(*)$ is contractible, so for $M = *$, the map has the form $\widetilde{\mathrm{Diff}}(*) \to \Omega^\infty(F_{h\mathbf{Z}/2})$. As $\pi_i \widetilde{\mathrm{Diff}}(*) \cong \Theta_{i+1}$, this provides invariants of exotic spheres.

[4] An english translation of [Wei86], which includes some corrections, is available at `https://www.maths.ed.ac.uk/~v1ranick/papers/weissex.pdf` Note that the references in [WW88] contain corrections to [Wei86] as well.

References

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