Pontryagin–Weiss classes

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(joint work with Alexander Kupers)

Pontryagin classes are certain characteristic classes of real vector bundles $\pi\colon E \to X$. There is one for every nonnegative integer $i \geq 0$, defined as the $2i$th Chern class of the complexification of the vector bundle up to a sign:

$$p_i(\pi) \coloneqq (-1)^i \cdot c_{2i}(\pi) \in \mathrm{H}^{4i}(X;\mathbf{Z}).$$

Two of their most important properties are:

(Stability) $p_i(\pi) = p_i(\pi \oplus \mathbf{R})$.

(Vanishing) $p_i(\pi) = 0$ for $i > \frac{\mathrm{rank}(\pi)}{.}2$

One of the historically most significant applications of Pontryagin classes is that, when applied to tangent bundles, they give rise to invariants of smooth manifolds that can be used to distinguish smooth structures on a given topological manifold. This fundamentally uses that Pontryagin classes are *integral* characteristic classes: when considered as rational characteristic classes, their values on tangent bundles of manifolds M are independent of the smooth structure. The reason for this is a combination of two facts: firstly, by work of Novikov, Kirby–Siebenmann, Sullivan, and others, there is a unique extension of the rational Pontryagin classes $p_i(\pi) \in \mathrm{H}^{4i}(X;\mathbf{Q})$ to stable characteristic classes of $\mathbf{R}^d$*-bundles*—fibre bundles whose fibres are homeomorphic to $\mathbf{R}^d$ (without a fibrewise linear structure). Secondly, by results of Milnor, Kister, and Mazur, every topological d-manifold M has a *topological tangent bundle* which is an $\mathbf{R}^d$-bundle that agrees with the underlying $\mathbf{R}^d$-bundle of the usual tangent bundle if M comes with a smooth structure.

By construction, these more general rational Pontryagin classes $p_i(\pi) \in \mathrm{H}^{4i}(X;\mathbf{Q})$ that are defined for $\mathbf{R}^d$-bundles still satisfy the above stability property, but it was unclear for a long time whether they also satisfy the vanishing property (c.f. [Sul05, p. 210]) until Weiss [Wei21] proved—to the surprise of many—that they do not:

Theorem (Weiss)**.** *There are pairs of integers (i,d) with $i > \frac{d}{2}$ such that there exists an $\mathbf{R}^d$-bundle over a sphere $\pi\colon E \to S^{4i}$ with $p_i(\pi) \neq 0$.*

In my talk I presented the following strengthening of Weiss' result from [KK24]:

Theorem A (Krannich–Kupers)**.** *For all pairs (i,d) with $d \geq 6$ and $i \geq 0$, there exists an $\mathbf{R}^d$-bundle over a sphere $\pi\colon E \to S^{4i}$ with $p_i(\pi) \neq 0$.*

Remark.

(1) Because of the stability property, the statement of Theorem A for all $d \geq 6$ follows from the case $d = 6$ by taking products with trivial bundles.
(2) Closely related to Theorem A, Galatius and Randal-Williams [GRW23] proved by a different approach that there are no universal relations between products of Pontryagin classes for $\mathbf{R}^d$-bundles when $d \geq 6$.

Theorem A can be shown to be equivalent to either of the following statements:

(1) The stabilisation map induced by taking products with Euclidean spaces

$$\mathrm{Top}(d) \longrightarrow \mathrm{Top} \coloneqq \mathrm{colim}_d \mathrm{Top}(d)$$

admits a rational section as long as $d \geq 6$. Here $\mathrm{Top}(d)$ is the topological group of homeomorphisms of $\mathbf{R}^d$ in the compact-open topology.

(2) Certain characteristic classes for smooth bundles with closed d-discs as fibres and a trivialisation of the boundary bundle which are defined in terms of Pontryagin classes and the Euler class (see [KRW21, 8.2.1]),

$$\mathrm{BDiff}_\partial(D^d) \longrightarrow \prod_{i \geq \lfloor \frac{d}{2} \rfloor} K(4i - d - 1, \mathbf{Q}), \tag{1}$$

can be detected for all $d \geq 6$ by bundles over spheres, i.e. the map (1) is surjective on rational homotopy groups.

Our strategy of proof for Theorem A is closely inspired by Weiss' original strategy, which relies on embedding calculus in the sense of Goodwillie–Weiss [Wei99, GW99] and Galatius–Randal-Williams' work on stable moduli spaces of manifolds [GRW17]. The most important additional ingredient in enhancing his strategy in order to prove Theorem A is the following result on the space of derived automorphisms of the rationalised little d-discs operad $E_d^{\mathbf{Q}}$:

Theorem B (Krannich–Kupers)**.** *For all $d \geq 2$, the double stabilisation map*

$$((-) \otimes \mathrm{id}_{E_2}^{\mathbf{Q}}) \colon \mathrm{BAut}(E_{d-2}^{\mathbf{Q}}) \longrightarrow \mathrm{BAut}(E_d^{\mathbf{Q}})$$

induced by taking Boardman–Vogt tensor products with $E_2^{\mathbf{Q}}$, is nullhomotopic.

Remark. Theorem B is optimal in the following sense:

(1) The single stabilisation $\mathrm{BAut}(E_{d-1}^{\mathbf{Q}}) \longrightarrow \mathrm{BAut}(E_{d-2}^{\mathbf{Q}})$ is not null in general: it can be shown to be nontrivial on $\mathrm{H}^{4n}(-;\mathbf{Q})$ for $d = 2n+1$.

(2) The non-rationalised version of the double stabilisation map $\mathrm{BAut}(E_{d-2}) \to \mathrm{BAut}(E_{d-2})$ is not null either: it factors the composition $\mathrm{BO}(d-2) \subset \mathrm{BO}(d) \to \mathrm{BG}(d) = \mathrm{BhAut}(S^{d-1})$ induced by block-inclusion of matrices and the usual $O(d)$-action on S^{d-1}. This composition induces the unstable J-homomorphism on homotopy groups, which is highly nontrivial.

References

[GRW17] S. Galatius and O. Randal-Williams, *Homological stability for moduli spaces of high dimensional manifolds. II*, Ann. of Math. (2) **186** (2017), no. 1, 127–204.

[GRW23] ———, *Algebraic independence of topological Pontryagin classes*, J. Reine Angew. Math. **802** (2023), 287–305.

[GW99] T. G. Goodwillie and M. Weiss, *Embeddings from the point of view of immersion theory. II*, Geom. Topol. **3** (1999), 103–118.

[KK24] M. Krannich and A. Kupers, *Pontryagin–Weiss classes and a rational decomposition of spaces of homeomorphisms*, 2024, in preparation.

[KRW21] M. Krannich and O. Randal-Williams, *Diffeomorphisms of discs and the second Weiss derivative of* BTop(−), 2021, arXiv:2109.03500.

[Sul05] D. P. Sullivan, *Geometric topology: localization, periodicity and Galois symmetry*, K-Monographs in Mathematics, vol. 8, Springer, Dordrecht, 2005, The 1970 MIT notes.

[Wei99] M. S. Weiss, *Embeddings from the point of view of immersion theory. I*, Geom. Topol. **3** (1999), 67–101.

[Wei21] M. S. Weiss, *Rational Pontryagin classes of Euclidean fiber bundles*, Geom. Topol. **25** (2021), no. 7, 3351–3424.