Pseudoisotopies of even discs, revisited

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The subject of *pseudoisotopy* or *h-cobordism theory* is the study of the homotopy type of the topological group

$$\mathrm{C}(M) = \{\phi\colon M \times [0,1] \xrightarrow{\cong} M \times [0,1] \mid \phi|_{M\times 0 \cup \partial M \times [0,1]} = \mathrm{id}\}$$

of *pseudoisotopies* of a compact smooth manifold M in the smooth topology. In my talk, I explained the following result from [5], which provides a p-local identification of this homotopy type in the case of a closed $2n$-dimensional disc D^{2n} in terms of the algebraic K-theory spectrum $K(\mathbb{Z})$ of the integers in a range up to roughly the dimension for primes p that are large with respect to the degree and the dimension.

Theorem. *For $n > 3$, there exists a zig-zag*

$$\mathrm{BC}(D^{2n}) \longrightarrow \cdot \longleftarrow \Omega_0^{\infty+1} K(\mathbb{Z})$$

whose maps are p-locally $\min(2n-4, 2p-4-n)$*-connected for primes p.*

So far, the relation of the homotopy type of spaces of pseudoisotopies with algebraic K-theory was studied via a combination of a stability result of Igusa [4] and foundational work of Waldhausen [7] and Waldhausen, Jahren, and Rognes [8]. The proof of the theorem above is independent of this approach and provides a new method to access spaces of pseudosisotopies of even-dimensional discs, which does not involve stabilising the dimension, yields a better range in many cases, and is *homological* (see [5] for an explanation). The most recent ingredient that goes into the proof of this result is Botvinnik and Perlmutter's computation of the stable homology of the moduli space of high-dimensional handlebodies [2].

Rationally and combined with a result of Randal-Williams [6] and Borel's work on the stable cohomology of arithmetic groups [1], our theorem results in the following partial computation of the rational homotopy groups of the group $\mathrm{Diff}_\partial(D^{2n+1})$ of diffeomorphisms of an odd-dimensional disc fixing the boundary pointwise.

Corollary. *There exists an isomorphism*

$$\pi_*\mathrm{BDiff}_\partial(D^{2n+1})\otimes\mathbb{Q} \cong K_{*+1}(\mathbb{Z})\otimes\mathbb{Q} \cong \begin{cases} \mathbb{Q} & \textit{if } * \equiv 0 \pmod 4 \\ 0 & \textit{otherwise} \end{cases} \quad \textit{for } 0 < * < 2n-5.$$

Remark.

(1) In a range of degrees up to approximately $2n/3$, these groups were previously known as a result of a computation of Farrell and Hsiang [3], who combined Waldhausen's approach to pseudoisotopy theory with the study of a certain involution, neither of which the proof of the corollary requires.

(2) From work of Watanabe [9] on the value of certain characteristic classes constructed by Kontsevich on disc bundles, one can deduce that the range in the corollary is optimal up to at most three degrees.

References

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