Moduli spaces of h-cobordisms of discs

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The classical s-cobordism theorem identifies the set of isomorphism classes of h-cobordisms on a compact connected smooth manifold M of dimension $d \geq 5$ with a quotient of the first algebraic K-group of $\mathbf{Z}[\pi_1 M]$—the *Whitehead group*

$$\mathrm{Wh}(\mathbf{Z}[\pi_1 M]) \coloneqq K_1(\mathbf{Z}[\pi_1 M])/\pm\pi_1 M.$$

This identification can be seen as a computation of the components of the *moduli space* $H(M)$ of h-cobordisms on M, and it turns out that the higher homotopy type of $H(M)$ is related to algebraic K-theory in a similar way: foundational work of Waldhausen [4] provides a canonical map

$$H(M) \to \Omega^\infty \mathrm{Wh}^{\mathrm{Diff}}(M)$$

to the infinite loop space of a spectral refinement of the Whitehead group—the *smooth Whitehead spectrum*

$$\mathrm{Wh}^{\mathrm{Diff}}(M) \coloneqq K(\mathbf{S}[\Omega M])/(\Sigma^\infty_+ M).$$

On path-components, this induces the isomorphism provided by the s-cobordism theorem, but this map is known to be more highly connected by a combination of two major results in the parametrised study of high-dimensional manifolds:

(i) Taking cylinders induces a stabilisation map $H(M) \to H(M \times [0,1])$ which is compatible with the map above and has been shown by Igusa [1] to be approximately $(d/3)$-connected, based on parametrised Morse theory.

(ii) The map $\mathrm{hocolim}_k H(M \times [0,1]^k) \to \mathrm{Wh}^{\mathrm{Diff}}(M)$ is an equivalence by Waldhausen–Jahren–Rognes' *stable parametrised h-cobordism theorem* [5].

In my talk, I explained a new approach to study the relation between $H(M)$ and $\mathrm{Wh}^{\mathrm{Diff}}(M)$ in the case $M = D^{2n}$. So far, it has led to the following.

Theorem A. *There exists a $(n-2)$-connected map $H(D^{2n}) \to \Omega^\infty \mathrm{Wh}^{\mathrm{Diff}}(*)$.*

Remark. In [2], based on a different strategy, it is shown that, as long as one is willing to invert primes that are large with respect to the dimension and the degree, then the map in Theorem A becomes twice as connected (see also [3]).

In contrast to (i) and (ii)—which Theorem A recovers in the case $M = D^{2n}$(with an improved range)—the proof of Theorem A does not involve stabilising the dimension, but instead relates $H(D^{2n})$ directly to algebraic K-theory. It relies on several ingredients of which some might be of independent interest, such as an analysis of the *moduli space of block-thickenings* of a finite complex, which is largely geometric, or a general homological vanishing result for the stable twisted homology of $\mathrm{BGL}(\mathbf{S}[\Omega M])$. I will explain the latter in a special case.

Stable twisted homology of $\mathrm{GL}_g(\mathbf{S})$. As $\pi_1\mathrm{BGL}_g(\mathbf{S}) \cong \mathrm{GL}_g(\mathbf{Z})$, a local system M_g over $\mathrm{BGL}_g(\mathbf{S})$ is a $\mathbf{Z}[\mathrm{GL}_g(\mathbf{Z})]$-module. Given a sequence $M_g \to M_{g+1}$ of compatible local systems, we can form the colimit

$$\mathrm{H}_*(\mathrm{BGL}(\mathbf{S}); M_\infty) = \mathrm{colim}_g \mathrm{H}_*(\mathrm{BGL}_g(\mathbf{S}); M_g) \tag{1}$$

and ask for conditions on the sequence M_g that ensure vanishing of (1). If M_g is constant, then (1) agrees with the homology of $\Omega_0^\infty K(\mathbf{S})$, so it can be very nontrivial. Another example is $M_g = \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}^g, \mathbf{Z}^g)$ in which case (1) does not vanish either; for instance $\mathrm{H}_0(\mathrm{BGL}(\mathbf{S}); M_\infty) = \mathrm{colim}_g \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}^g, \mathbf{Z}^g)_{\mathrm{GL}_g(\mathbf{Z})}$ does not. Note however that for $M_g = \mathbf{Z}^g$, the analogous group $\mathrm{H}_0(\mathrm{BGL}(\mathbf{S}); M_\infty) = \mathrm{colim}_g (\mathbf{Z}^g)_{\mathrm{GL}_g(\mathbf{Z})}$ does vanish and this is no coincidence: the sequence $M_g = \mathbf{Z}^g$ is part of a class of sequences M_g that are induced by a an abelian-group valued functor M on the category $\mathrm{P}(\mathbf{Z})$ of finitely generated projective modules via

$$M_g \coloneqq M(\mathbf{Z}^g) \xrightarrow{M(\mathbf{Z}^g \subset \mathbf{Z}^g \oplus \mathbf{Z})} M(\mathbf{Z}^{g+1}) \eqqcolon M_{g+1},$$

where $\mathrm{GL}_g(\mathbf{Z})$ acts by functoriality. Such a functor is called *reduced* if $M(0) = 0$ and it is called *analytic* if it a colimit of polynomial functors. A simplified version of the homological vanishing result that goes in the proof of Theorem A shows that *any* sequence M_g that extends to such a functor has vanishing stable homology.

Theorem B. *For an abelian-group valued functor M on $\mathrm{P}(\mathbf{Z})$ that is reduced and analytic, the stable homology $\mathrm{H}_*(\mathrm{BGL}(\mathbf{S}); M_\infty)$ vanishes.*

References

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