Stability of concordance embeddings

Manuel Krannich
(joint work with Thomas Goodwillie and Alexander Kupers)

Concordance embeddings and diffeomorphisms. Fix a smooth d-manifold M and a compact submanifold $P \subset M$ meeting ∂M transversely. Writing $I \coloneqq [0,1]$, a *concordance embedding* of P into M is a smooth embedding

$$e\colon P \times I \hookrightarrow M \times I$$

that satisfies $e^{-1}(M \times \{i\}) = P \times \{i\}$ for $i = 0, 1$ and agrees with the inclusion $P \times I \subset M \times I$ in a neighbourhood of $P \times \{0\} \cup (P \cap \partial M) \times I$. The space of such embeddings, equipped with the smooth topology, is denoted $\mathrm{CE}(P, M)$. In the case $P = M$, every concordance embedding is in fact a *concordance diffeomorphism*, that is a diffeomorphism of $M \times I$ that is the identity on a neighbourhood of $M \times \{0\} \cup \partial M \times I$. One writes $\mathrm{C}(M)$ for the space of concordance diffeomorphisms. $\mathrm{C}(M)$ and $\mathrm{CE}(P, M)$ are closely related: a concordance diffeomorphism of M yields by restriction to $P \times I \subset M \times I$ a concordance embedding of P into M, and on the level of spaces, this observation leads to a fibre sequence of the form

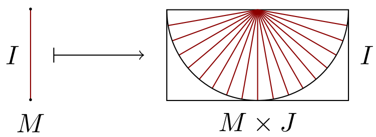
$$\mathrm{C}(M \backslash \nu(P)) \longrightarrow \mathrm{C}(M) \longrightarrow \mathrm{CE}(P, M) \tag{1}$$

where $\nu(P) \subset M$ is an open tubular neighbourhood of P.

Stabilisation. Concordance embeddings can be stabilised: there is a map

$$\sigma\colon \mathrm{CE}(P, M) \longrightarrow \mathrm{CE}(P \times J, M \times J)$$

by taking products with $J \coloneqq [-1, 1]$ and bending the result appropriately to make it satisfy the conditions for concordance embeddings, schematically like this:



It was Igusa [6] who, building Hatcher's work [5], showed that this stabilisation map is at least about $d/3$-connected—one of the key ingredients in studying manifolds and their diffeomorphism groups via surgery theory and pseudoisotopy theory. He phrased his result for concordance diffeomorphisms (i.e. the case $P = M$), but the version for general concordance embeddings follows from this, using (1).

The stability theorem. One consequence of the work with T. Goodwillie and A. Kupers that the talk was about is that, under a certain assumption on $P \subset M$, the stabilisation map is significantly more connected than the known $d/3$-bound. This "certain assumption" is a requirement on the *handle dimension* of the inclusion $P \subset M$, which is the minimal number p so that P can be built from a closed collar on $P \cap \partial M$ by attaching handles of index $\leq p$.

Theorem A (Goodwillie–Krannich–Kupers)**.** *If the handle dimension p of $P \cap \partial M \subset P$ satisfies $p \leq d-3$, then the stabilisation map*

$$\sigma\colon \mathrm{CE}(P, M) \longrightarrow \mathrm{CE}(P \times J, M \times J)$$

is $(2d-p-5)$-connected.

Remark. The case $P = *$ was previously known from work of G. Meng [8].

An application. One use case of Theorem A is the following: via the fibre sequence (1) for various choices of submanifolds $P \subset M$, this result puts one in the position to transfer information on the stabilisation map for concordance diffeomorphisms of a specific manifold M (for example lower or upper connectivity bounds) to other manifolds. For instance, together with O. Randal-Williams, I computed as part of [7] the rationalised relative homotopy groups $\pi_*(\mathrm{C}(M \times J), \mathrm{C}(M)) \otimes \mathbb{Q}$ for high-dimensional closed discs $M = D^d$ in a range of degrees beyond that in which these groups vanish, and Theorem A allows for an extension of this computation from discs to any high-dimensional simply-connected spin manifold M [3, Corollary C].

The multirelative stability theorem. Theorem A is a special case of our main theorem, which is a more general "multirelative" version. Interestingly, the proof of the more general version involves an induction that would fail if one tried to only prove the special case stated as Theorem A. Said differently, the more general version is not only more general, but also necessary (at least for our proof).

In addition to the submanifold $P \subset M$, the statement of the multirelative version involves compact submanifolds $Q_1, \ldots, Q_r \subset M$ that are pairwise disjoint as well as disjoint from P. Writing $M_S \coloneqq M \backslash \cup_{i \notin S} Q_i$ for subsets $S \subset \underline{r}$ of $\underline{r} \coloneqq \{1, \ldots, r\}$, there are inclusions $\mathrm{CE}(P, M_S) \subset \mathrm{CE}(P, M_{S'})$ whenever $S \subset S'$. This enhances the space $\mathrm{CE}(P, M)$ to an *r-cube*—a space-valued functor on the poset of subsets of $\underline{r}$. This functor $\underline{r} \supset S \mapsto \mathrm{CE}(P, M_S)$ is denoted by $\mathrm{CE}(P, M_\bullet)$. Note that the value at the empty set recovers $\mathrm{CE}(P, M)$. Defined suitably, the stabilisation map extends to a *map of r-cubes* (meaning, a natural transformation)

$$\sigma\colon \mathrm{CE}(P, M_\bullet) \longrightarrow \mathrm{CE}(P \times J, (M \times J)_\bullet)$$

whose target is the r-cube involving the submanifolds $Q_i \times J \subset M \times J$.

Our multirelative stability theorem (which specialises to Theorem A by setting $r = 0$) is an estimate on the connectivity of this map of r-cubes in terms of the handle dimensions p and q_i of the inclusions $\partial M \cap P \subset P$ and $\partial M \cap Q_i \subset Q_i$.

Theorem B (Goodwillie–Krannich–Kupers)**.** *If the handle dimensions satisfy $p \leq d-3$ and $q_i \leq d-3$ for all i, then the stabilisation map of r-cubes*

$$\sigma\colon \mathrm{CE}(P, M_\bullet) \longrightarrow \mathrm{CE}(P \times J, (M \times J)_\bullet)$$

is $(2d-p-5+\sum_{i=1}^{r}(d-q_i-2))$-connected.

Here are the relevant definitions: an r-cube $X_\bullet$ is k-cartesian if the natural map $X_\emptyset \to \mathrm{holim}_{\emptyset \neq S \subset \underline{r}} X_S$ is k-connected in the usual sense, and a map of r-cubes $X_\bullet \to Y_\bullet$ is k-connected if a certain $(r+1)$-cube is k-cartesian, namely the $(r+1)$-cube that maps $S \subset \underline{r+1}$ to X_S if $S \subset \underline{r}$ and to $Y_{S \backslash \{r+1\}}$ otherwise.

Analyticity and calculus. Theorem B is in line with previous multirelative connectivity results in geometric topology. Let me mention the two that are the closest to Theorem B (incidentally both used in our proof). The statement involves the quantity $\Sigma \coloneqq \sum_{i=1}^{r}(d-q_i-2)$ and the space $\mathrm{E}(P,M)$ of ordinary embeddings $P \hookrightarrow M$ that agree with the inclusion in a neighbourhood of $P \cap \partial M$.

(a) If $p, q_i \leq d-3$, then the r-cube $\mathrm{CE}(P, M_\bullet)$ is $(d-p-2+\Sigma)$-cartesian, by [1].
(b) If $p, q_i \leq d-3$, then the r-cube $\mathrm{E}(P, M_\bullet)$ is $(1-p+\Sigma)$-cartesian, by [2].

These two results as well as Theorem B may be viewed as analyticity results in the sense of Goodwillie–Weiss' *manifold calculus* [4, 9] for functors on a suitable poset category of compact submanifolds of M, namely the functors sending $P \subset M$ to $\mathrm{E}(P,M)$, $\mathrm{CE}(P,M)$, or $\mathrm{hofib}(\mathrm{CE}(P,M) \to \mathrm{CE}(P \times I, M \times I))$ respectively. There is also an intriguing connection to the approach to studying diffeomorphism groups by means of Weiss' *orthogonal calculus* [10], waiting to be explored.

References

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