PROBLEMS FROM THE HCM WORKSHOP ON AUTOMORPHISMS OF MANIFOLDS

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Abstract. These are problems and questions posed by speakers of the workshop "Automorphisms of Manifolds" which took place in September 2019 at the Hausdorff Center for Mathematics in Bonn.

1. Alexander Kupers

1.1. **The homotopy type of the topological bordism category in dimension 4.** Let $\theta\colon B \to \mathrm{BTop}(d)$ be a tangential structure of topological $\mathbf{R}^d$-bundles. By the main result of [GLK18], there exists a weak equivalence

$$\Omega\,\mathrm{BCob}_d^{\mathrm{Top},\theta} \simeq \Omega^\infty \mathbf{MT}\theta \quad \text{for } d \neq 4, \tag{1}$$

where $\mathrm{Cob}_d^{\mathrm{Top},\theta}$ is the topological bordism category of d-dimensional topological manifolds with θ-structure and $\mathbf{MT}\theta$ the Thom spectrum of the inverse of the universal $\mathbf{R}^d$-bundle over $\mathrm{BTop}(d)$.

Problem 1.1. *Establish the weak equivalence (1) in the case $d = 4$.*

Remark. The corresponding result for smooth manifolds by Galatius–Madsen–Tillmann–Weiss [GTMW09] is also valid in dimension $d = 4$. The assumption $d \neq 4$ in [GLK18] is due to the use of smoothing theory to deduce the topological case from the smooth one.

1.2. **Stable homology of the moduli space of handlebodies in dimensions 5 and 7.** Denote by $\mathrm{Diff}_{D^{2n}}(V_g)$ the topological group of diffeomorphisms of a handlebody $V_g = \natural^g D^{n+1} \times S^n$ fixing a chosen embedded $2n$-disc in the boundary pointwise. Taking boundary connected sums with $D^{n+1} \times S^n$ induces a stabilisation map $\mathrm{BDiff}_{D^{2n}}(V_g) \to \mathrm{BDiff}_{D^{2n}}(V_g)$ and scanning provides a canonical map

$$\mathrm{hocolim}_g\, \mathrm{BDiff}_{D^{2n}}(V_g) \longrightarrow Q_0(\mathrm{BSO}(2n+1)\langle n\rangle_+), \tag{2}$$

where $\mathrm{BSO}(2n+1)\langle n\rangle$ denotes the n-connected cover of $\mathrm{BSO}(2n+1)$. In dimensions $2n + 1 \geq 9$, this is a homology equivalence by the main result of [BP17].

Question 1.2. *Is (2) also a homology equivalence in dimensions $2n + 1 = 5, 7$?*

Remark. For $2n + 1 = 7$ the answer is likely "yes" as there are only two steps in the proof of Botvinnik–Perlmutter which exclude dimension 7: a higher-dimensional version of the half Whitney trick and an connectivity estimate in its application (both in [BP17, Proposition 5.10]).

1.3. **Configuration space integrals.** There are various different definitions of configuration space integrals in the literature, tailored to specific applications. However, we currently lack a conceptual (and more homotopy-theoretical) construction which satisfies a universal property. This would allow for easy comparison to other definitions, and might help answer one or more of the following questions:

(1) Can configuration space integrals described more axiomatically?

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(2) Over which coefficients can they be defined? $\mathbf{R}$, $\mathbf{Z}$, or maybe even $\mathbb{S}$?
(3) For what manifolds can they be defined? Smooth manifolds, PL manifolds, topological manifolds? What role do tangential structures such as framings play?
(4) Is there a relative version?
(5) Is there a formula for the effect of gluing or surgery?
(6) Which are other good properties do they have?

Problem 1.3. *Give a "good" description of configuration space integrals and answer some of the questions above.*

1.4. **Isotopy extension for embedding calculus.** Let M and N be compact manifolds of the same dimension and $P \subset M$ a compact submanifold of codimension zero. Denote by $\iota\colon P \to M$ the inclusion map. By isotopy extension, there is a weak equivalence

$$\mathrm{Emb}_{\partial P}(M \setminus \mathrm{int}(P), N \setminus \mathrm{int}(\iota(P))) \simeq \mathrm{hofib}_\iota[\mathrm{Emb}(M,N) \to \mathrm{Emb}(P,N)].$$

Under appropriate conditions on the (relative) handle dimensions and dimensions of P, M, and N, the embedding calculus tower is known to converge for each of these terms, so we also have a weak equivalence

$$T_\infty \mathrm{Emb}_{\partial P}(M \setminus \mathrm{int}(\iota(P)), N \setminus \mathrm{int}(P)) \simeq \mathrm{hofib}_\iota[T_\infty \mathrm{Emb}(M,N) \to T_\infty \mathrm{Emb}(P,N)].$$

Question 1.4. *Is there such a weak equivalence even when the embedding calculus tower for one of the terms is not known to converge?*

2. Oscar Randal-Williams

2.1. **Embedding calculus for topological manifolds.** Using smoothing theory, one can deduce convergence of the embedding calculus tower for topological embeddings of smoothable topological manifolds satisfying suitable conditions. So far, there is no proof in the topological category which also applies to non-smoothable topological manifolds.

Problem 2.1. *Under suitable assumptions, give an intrinsic proof of convergence of the embedding calculus tower in the topological category.*

2.2. **Stability for homeomorphisms of topological 4-manifolds.** Let M be a topological simply connected 4-manifold with nonempty boundary.

Question 2.2. *Does* $\mathrm{BHomeo}(M)$ *satisfy homological stability with respect to taking connected sums with* $S^2 \times S^2$*?*

Remark. The corresponding result in dimensions $2n \geq 6$ was proved by Kupers [Kup15].

2.3. **Characteristic classes and embedding calculus.** Let M be a $2n$-dimensional closed smooth oriented manifold. Consider the delooped Taylor tower of its group of orientation-preserving diffeomorphisms,

$$\begin{tikzcd}
 & \mathrm{BT_\infty Diff(M)} \arrow[d] \\
 & \cdots \arrow[d] \\
\mathrm{BDiff}(M) \arrow[uur] \arrow[ur] \arrow[r] \arrow[dr] \arrow[ddr] & \mathrm{BT_1Diff(M)} \simeq \mathrm{BhAut(TM)} \arrow[d] \\
 & \mathrm{BhAut(M)} \arrow[d] \\
 & \mathrm{BO}(\mathrm{H}_n(M;\mathbf{Z})/\mathrm{tors}, \lambda),
\end{tikzcd}$$

extended to the bottom by the space $\mathrm{hAut}(M)$ of orientation-preserving homotopy automorphisms of M and the group $\mathrm{O}(\mathrm{H}_n(M;\mathbf{Z})/\mathrm{tors}, \lambda)$ of automorphisms of the torsion free quotient of $\mathrm{H}_n(M;\mathbf{Z})$ preserving the intersection form λ.

As explained for instance in [RW19], there are classes $\sigma_j \in \mathrm{H}^j(\mathrm{BO}(\mathrm{H}_n(M;\mathbf{Z})/\mathrm{tors},\lambda);\mathbf{Q})$ for $j \equiv 2n \mod 4$, constructed in terms of signatures of symmetric forms, which when pulled back to $\mathrm{BDiff}(M)$ can be expressed in terms of generalised Miller–Morita–Mumford classes as

$$\sigma_{4i-2n} = \kappa_{\mathcal{L}_i} := \int_\pi \mathcal{L}_i(T_\pi E), \tag{3}$$

for $\mathcal{L}_i$ the Hirzebruch $\mathcal{L}$-classes and $\pi : E \to \mathrm{BDiff}(M)$ the universal oriented smooth M-bundle. This was first proved by Atiyah using the family index theorem, but can also be deduced from the Hirzebruch signature theorem, cf. [RW19].

Like all generalised Miller–Morita–Mumford classes, the classes $\kappa_{\mathcal{L}_i}$ may be defined on the space $\mathrm{BhAut}(TM)$, and one may ask whether the identity $\sigma_{4i-2n} = \kappa_{\mathcal{L}_i}$ already holds on this space: it is easy to see that it does not, by direct calculation with e.g $M = S^{2n}$. One may then wonder whether embedding calculus arranges to fix this defect, or not.

Question 2.3. *Is it true that $\sigma_{4i-2n} \neq \kappa_{\mathcal{L}_i} \in \mathrm{H}^{4i-2n}(\mathrm{BT}_\infty\mathrm{Diff}(M);\mathbf{Q})$?*

Remark. A positive answer to this question would in particular show that the Taylor tower of $\mathrm{Diff}(M)$ does not converge.

The identity (3) has consequences for relations among generalised Miller–Morita–Mumford classes: as $\mathrm{O}(\mathrm{H}_n(M;\mathbf{Z})/\mathrm{tors},\lambda)$ is an arithmetic group it has finite vcd and so the cohomology classes σ_j are nilpotent, hence the $\kappa_{\mathcal{L}_i} \in \mathrm{H}^{4i-2n}(\mathrm{BDiff}(M);\mathbf{Q})$ are also nilpotent. There are several other methods for obtaining relations among generalised Miller–Morita–Mumford classes [Gri17, GGRW17, RW18] but they are largely homotopy-theoretic and apply already on the spaces $\mathrm{BhAut}(TM)$ or $\mathrm{B}\widetilde{\mathrm{Diff}}(M)$. Thus they do not really use the smooth fibre bundle structure, apart from perhaps taking (3) as input.

Question 2.4. *Assuming the answer to Question 2.3 is positive, are there other relations among generalised Miller–Morita–Mumford classes which hold on $\mathrm{BDiff}(M)$, but not on $\mathrm{BT}_\infty\mathrm{Diff}(M)$?*

Unrelated to embedding calculus, one may ask the analogous question for $\mathrm{B}\widetilde{\mathrm{Diff}}(M)$.

Question 2.5. *Are there relations among generalised Miller–Morita–Mumford classes which hold on $\mathrm{BDiff}(M)$ but not on $\mathrm{B}\widetilde{\mathrm{Diff}}(M)$?*

Remark. One should of course exclude the obvious relations stemming from the identities $\{e^2 = p_n, p_i = 0 \text{ for } i > n\}$ among characteristic classes of $2n$-dimensional vector bundles.

Remark. The identity $\sigma_{4i-2n} = \kappa_{\mathcal{L}_i}$ holds in $\mathrm{H}^{4i-2n}(\mathrm{B}\widetilde{\mathrm{Diff}}(M);\mathbf{Q})$, by [RW19].

2.4. **Stable characteristic classes of block bundles.** Berglund–Madsen [BM14] have computed the ring

$$\mathrm{H}^*(\mathrm{hocolim}_g\, \mathrm{B}\widetilde{\mathrm{Diff}}(\sharp^g(S^n\times S^n), D^{2n});\mathbf{Q})$$

of stable rational characteristic classes of block bundles with fibre $\sharp^g(S^n\times S^n)$, relative to a disc $D^{2n} \subset \sharp^g(S^n\times S^n)$ as a free graded commutative algebra generated by the pull backs of the classes σ_i (see Section 2.3) and certain classes $\widetilde{\kappa}^{\xi}_{p_{i_1},\dots p_{i_s}}$ associated to Pontryagin classes and $\xi \in \mathrm{H}^*(\mathrm{BA}_{h,s};\mathbf{Q})$, at least up to taking associated graded with respect to a multiplicative filtration. Here $A_{h,s}$ denotes the discrete group of homotopy classes of homotopy automorphisms of $\vee^h S^1$ fixing s marked points.

Problem 2.6. *Find a geometric description of the classes $\widetilde{\kappa}^{\xi}_{p_{i_1},\dots p_{i_s}}$ as characteristic classes of block bundles.*

3. Michael Weiss

3.1. **Pontryagin classes and automorphisms of the little disc operad.** Using configuration categories as in [BdBW18], one sees that there is a canonical map $\mathrm{BTop}(d) \to \mathrm{BhAut}^h(\mathcal{E}_d)$, where $\mathrm{Top}(d)$ is the topological group of homeomorphisms of $\mathbf{R}^d$ and $\mathrm{hAut}^h(\mathcal{E}_d)$ is the topological monoid of derived homotopy automorphisms of the little d-discs operad.

Question 3.1. *Do the rational Pontryagin classes $p_i \in \mathrm{H}^{4i}(\mathrm{BTop}(d); \mathbf{Q})$ pull back from $\mathrm{H}^{4i}(\mathrm{Baut}^h(\mathcal{E}_d); \mathbf{Q})$?*

Remark. If one replaces the operad $\mathcal{E}_d$ by its truncated version $\mathcal{E}_d^{\leq r}$ for any r, or by its rationalisation $(\mathcal{E}_d)_{\mathbf{Q}}$, the answer is "no".

Remark. This question is closely related to Question 2.3.

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